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Physics from the *G*-Bundle Viewpoint

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1. INTRODUCTION

It is generally agreed that in quantum mechanics, the observables of a physical system are represented by self-adjoint operators on a Hilbert space. The possible values of measurements of an observable are eigenvalues of the associated operator. Although all infinite-dimensional separable Hilbert spaces are isomorphic, obviously not all physical systems are the same. A physical system has some classical potential (or more generally, some geometry) associated to it that helps one select the correct self-adjoint quantum mechanical operators for certain classical observables. The most important observables (e.g., energy) are typically differential operators that are *naturally* associated with the geometry of the classical picture; and they operate on functions defined on the configuration space that supports the geometry. It is the domain of geometric quantization to determine precisely what operators go with what classical observables. Here we will be concerned with studying some particularly natural differential operators (and their spectra) associated with systems that are classically modeled on principal fiber bundles which are equipped with a metric tensor on the base as well as a connection (i.e., gauge potential). Arguments are given that suggest that the eigenvalues of these operators are possible values of the kinetic energy of particles that respond to the gauge potential when the dimension of the base is three, and mass (squared) spectra of such particles when the base is four dimensional (e.g., space-time, or a Euclidean analog). These operators, which are essentially covariant Laplace operators obtained by the minimal coupling prescription, have become increasingly relevant with the growing realization that all interactions may ultimately be derived from those of the gauge type (e.g., the current times current Fermi interac-

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tion is now believed to be mediated by the intermediate vector bosons of a quantized non-Abelian gauge field in the Weinberg–Salam model).

In Section 2, principal G bundles are defined (assuming that the reader has a basic familiarity with manifolds and Lie groups). The primordial nontrivial Hopf bundle is discussed in detail. Moreover, connections (i.e., gauge potentials) and their curvatures (i.e., field strengths) are introduced; computations are carried out for the Hopf bundle, and we digress a bit, in discussing Chern numbers and Dirac monopoles.

In Section 3, we define and discuss some fundamental natural differential operators on function spaces intrinsically associated with the geometry of principal fiber bundles, connections, and metric tensors on the base. In particular, the relation between these operators and the ones encountered by means of minimal coupling is brought out. To properly discuss covariant Dirac operators (which appear to be more physically relevant) from a coordinate-free viewpoint would require the introduction of spin structures (e.g., see Chichilnisky, 1972). Since the main ideas of this paper carry over, we postpone the consideration of Dirac operators to a future publication.

In Section 4, we review the essential properties of Kaluza–Klein-type metrics on principal bundles, and geometrically interpret the covariant Laplacian of Section 3 in terms of a horizontal Laplacian of a Kaluza–Klein metric. This enables us to compute the spectrum of the covariant Laplacian on wave functions (for charged particles in the field of a monopole) in a straightforward manner. The eigenfunctions are identified as the monopole harmonics of Wu and Yang (1976).

In Section 5, we show how the degeneracy of the eigenvalues of covariant Laplacians is related to the group of automorphisms of the principal bundle that preserve the connection and the metric on the base. Moreover, we give the first-order perturbation formulas that describe how a degenerate eigenvalue is split under a change of connection and a change of metric on the base. These formulas generalize those associated with the Zeeman and Stark effects to the case of general (possibly, non-Abelian) gauge fields.

In Section 6, we give a detailed proof that the group of gauge transformations preserving a connection (the group responsible for degeneracy of eigenvalues) is isomorphic to the centralizer of the holonomy group of the connection. It is hypothesized that mass splittings within observed multiplets of particles are due to the small deviation of the hypothetical connection (of the generalized Kaluza–Klein universe) from non-generic connections with smaller holonomy groups. This seems to be more natural than the usual procedure of artificially introducing masses and mass splittings by adding Yukawa terms by hand to the Lagrangian and invoking the Higgs mechanism.

2. PRINCIPAL BUNDLES AND GAUGE POTENTIALS

Without loss of generality, we assume that G is a Lie group of matrices [e.g., $SU(N)$]. Let P be a manifold on which G acts smoothly and freely to the right. The action of $g \in G$ is denoted by $R_g\colon P \to P$; $p \mapsto R_g(p) = pg$. We assume that the space of orbits P/G is identified with a smooth (C^∞) manifold M, in such a way that the projection $\pi\colon P \to P/G \cong M$ is smooth and *locally trivial*; i.e., for all $x \in M$, there is a neighborhood U of x such that there is a diffeomorphism (smooth map with smooth inverse) $T\colon \pi^{-1}(U) \to U \times G$ of the form $T(p) = (\pi(p), s(p))$ which preserves the group actions, in the sense that $s(pg) = s(p)g$. In this case, we say "$\pi\colon P \to M$ is a (principal) G bundle." The bundle is *trivial*, if the neighborhood U above can be taken to be all of M; i.e., we have a $T\colon P \cong M \times G$ which preserves the G actions. A *local section* (or *choice of gauge*) is a map $\sigma\colon U \to P$ (U an open subset of M) such that $\pi(\sigma(y)) = y$ for all $y \in U$. Such a σ can be defined on all of M only when the bundle is trivial; given $\sigma\colon M \to P$, define $T\colon P \to M \times G$ by $T(\sigma(y)g) = (y, g)$. A choice of gauge essentially selects a way of identifying fibers $\pi^{-1}(y)$ with the group G.

Example 1. Let G be a Lie subgroup of $\bar{G}$. Take $P = \bar{G}$ with $R_g(\bar{g}) = \bar{g}g$ for $g \in G$, $\bar{g} \in \bar{G}$. Then $\pi\colon \bar{G} \to \bar{G}/G$ is a principal G bundle. In particular, take

$$\bar{G} = SU(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

and

$$G = U_3(1) = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} (= e^{i\theta\sigma_3}) : \theta \in \mathbb{R} \right\}$$

Note that $U_3(1) \cong U(1)$ via $e^{i\theta\sigma_3} \leftrightarrow e^{i\theta}$, while $SU(2)$ may be identified with the 3-sphere of unit quaternions via the isomorphism $SU(2) \cong S^3$ given by $\begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \leftrightarrow (a + jb)$; note that this sends $U_3(1)$ to $U(1)$. Hence, we have an "equivalence of bundles"

$$\begin{array}{ccc} SU(2) & \leftrightarrow & S^3 \\ \downarrow & & \downarrow \\ SU(2)/U_3(1) & \leftrightarrow & S^3/U(1) \end{array}$$

If we can show $SU(2)/U_3(1) \cong S^2 \equiv$ unit sphere in $\mathbb{R}^3$, then these (Hopf) bundles are nontrivial, since $S^3 \not\cong S^2 \times S^1$ topologically. To this end, define the customary $2-1$ homomorphism $H: SU(2) \to SO(3)$, using the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Namely, for $\mathbf{r} = (x, y, z)$ and

$$\mathbf{r} \cdot \boldsymbol{\sigma} = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix}$$

define $H(A)$ for $A \in SU(2)$ by $H(A)(\mathbf{r}) \cdot \boldsymbol{\sigma} = A(\mathbf{r} \cdot \boldsymbol{\sigma})A^*$. Now, define $\pi: SU(2) \to S^2$ by $\pi(A) = H(A)\mathbf{e}_3$, where $\mathbf{e}_3 = [0, 0, 1]^T$. Since $H(Ae^{i\theta\sigma_3})(\mathbf{e}_3) \cdot \boldsymbol{\sigma} = Ae^{i\theta\sigma_3}\sigma_3 e^{-i\theta\sigma_3}A^* = A\sigma_3 A^* = H(A)(\mathbf{e}_3) \cdot \boldsymbol{\sigma}$, we have $\pi(Ae^{i\theta\sigma_3}) = \pi(A)$, whence π induces a map $SU(2)/U_3(1) \to S^2$ which can be shown to be bijective.

Let G be the matrix Lie algebra of the matrix Lie group G. For $B \in \mathsf{G}$, let B^* be the vector field on P given by $B_p^* = (d/dt)(pe^{tB})|_{t=0}$. A *connection* for the G bundle $\pi: P \to M$ is a G-valued 1-form ω on P such that (1) $\omega(B^*) = B$ for all $B \in \mathsf{G}$, and (2) $\omega_{pg}(R_{g*}X) = g^{-1}\omega_p(X)g$ for all $X \in T_pP$, $p \in P$, $g \in G$, where $R_{g*}: T_pP \to T_{pg}P$ is the differential of $R_g: P \to P$ at p.

Note: If $\tau: U \to P$ is a local section (or choice of gauge), then the pull-back $\tau^*\omega$ is a G-valued 1-form on $U \subset M$ which is called a *gauge potential*; $\tau^*\omega$ depends on τ and is defined on all of M only if $U = M$ (i.e., $\pi: P \to M$ is trivial). Singularities are encountered when one tries to extend local sections and gauge potentials to all of M, when the bundle is nontrivial; this is the origin of the famous Dirac string in the theory of monopoles.

Example 2. Recall the G bundle $\pi: \bar{G} \to \bar{G}/G$ of Example 1. For $\bar{g} \in \bar{G}$ and $\bar{B} \in \bar{\mathsf{G}}$, note that $\bar{g}\bar{B} = (d/dt)\bar{g}e^{t\bar{B}}|_{t=0} \in T_{\bar{g}}\bar{G}$. Let $\bar{\omega}$ be the $\bar{\mathsf{G}}$-valued 1-form on $\bar{G}$ given by $\bar{\omega}_{\bar{g}}(\bar{g}\bar{B}) = \bar{B}$; $\bar{\omega}$ is called the *Maurer–Cartan form* of $\bar{G}$. Suppose $\bar{\mathsf{G}} = \mathsf{G} \oplus \mathsf{M}$, where $g\mathsf{M}g^{-1} = \mathsf{M}$, for all $g \in G$. Let $\pi_{\mathsf{G}}: \mathsf{G} \oplus \mathsf{M} \to \mathsf{G}$ be the projection and set $\omega = \pi_{\mathsf{G}} \circ \bar{\omega}$; one easily checks that ω is a connection for the G bundle $\pi: \bar{G} \to \bar{G}/G$.

In particular, for $\bar{G} = SU(2)$, $G = U_3(1)$, we have $\mathsf{G} = \{iz\sigma_3 : z \in \mathbb{R}\}$, $\mathsf{M} = \{i(x\sigma_1 + y\sigma_2) : x, y \in \mathbb{R}\}$. For $A \in SU(2)$, $T_A SU(2) = \{Ai\mathbf{r} \cdot \boldsymbol{\sigma} | \mathbf{r} \in \mathbb{R}^3\}$ and $\bar{\omega}(Ai\mathbf{r} \cdot \boldsymbol{\sigma}) = i\mathbf{r} \cdot \boldsymbol{\sigma}$, while $\omega(Ai\mathbf{r} \cdot \boldsymbol{\sigma}) = iz\sigma_3$.

Given a connection ω on P and $X \in T_pP$, we can split X into its horizontal and vertical parts, say, $X = X^H + X^V$, where $\pi_* X^V = 0$ and

$\omega(X^{H})=0$. For a vector space W, we denote the space of W-valued k forms on P by $\Lambda^{k}(P,W)$. Define $D^{\omega}:\Lambda^{k}(P,W)\rightarrow\Lambda^{k+1}(P,W)$ by $(D^{\omega}\alpha)(X_1,\ldots,X_{k+1})\equiv(d\alpha)(X_1^{H},\ldots,X_{k+1}^{H})$, where d is the usual exterior derivative operator. The *curvature* of $\omega\in\Lambda^{1}(P,\mathrm{G})$ is defined to be $\Omega\equiv D^{\omega}\omega\in\Lambda^{2}(P,\mathrm{G})$. Alternatively (See [1] or [8] for proof.), $\Omega=d\omega+(1/2)[\omega,\omega]$, where $(1/2)[\omega,\omega](X,Y)=(1/2)([\omega(X),\omega(Y)]-[\omega(Y),\omega(X)])=[\omega(X),\omega(Y)]$. For $g\in G$, one can prove $R_g^{*}\Omega=g^{-1}\Omega g$, where R_g^{*} is "pull-back" of forms by R_g. If G is Abelian, then $R_g^{*}\Omega=\Omega$, and for a choice of gauge $\tau:U\rightarrow P$ the *field strength* $\equiv\tau^{*}\Omega$ is independent of τ; for G non-Abelian this is no longer true in general.

Example 3. The Maurer–Cartan form $\bar{\omega}$ on $\bar{G}$ of Example 2 satisfies $d\bar{\omega}=-(1/2)[\bar{\omega},\bar{\omega}]$. Hence, the connection $\omega=\pi_{\mathrm{G}}\circ\bar{\omega}$ for the G bundle $\pi:\bar{G}\rightarrow\bar{G}/G$ has curvature Ω given at $\bar{g}\in\bar{G}$ (in the notation of Example 2) by

$$\Omega(\bar{g}\bar{B},\bar{g}\bar{C})=-\pi_{\mathrm{G}}\left[\pi_{\mathrm{M}}\bar{B},\pi_{\mathrm{M}}\bar{C}\right]$$

where $\pi_{\mathrm{G}}:\mathrm{G}\oplus\mathrm{M}\rightarrow\mathrm{G}$ and $\pi_{\mathrm{M}}:\mathrm{G}\oplus\mathrm{M}\rightarrow\mathrm{M}$ are the projections.

When $\bar{G}=SU(2)$ and $G=U_3(1)$, we use the identity $[\mathbf{r}\cdot\boldsymbol{\sigma},\mathbf{r}'\cdot\boldsymbol{\sigma}]=2i(\mathbf{r}\times\mathbf{r}')\cdot\boldsymbol{\sigma}$ to obtain

$$\Omega(Ai\mathbf{r}\cdot\boldsymbol{\sigma},Ai\mathbf{r}'\cdot\boldsymbol{\sigma})=2i(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{e}_3\sigma_3 \qquad (*)$$

Recalling that $\pi:SU(2)\rightarrow S^2$ is given by $\pi(A)\cdot\boldsymbol{\sigma}=(H(A)\mathbf{e}_3)\cdot\boldsymbol{\sigma}=A\sigma_3A^{*}$, we calculate

$$\begin{aligned}\pi_{*}(Ai\mathbf{r}\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\sigma}&=(d/dt)\pi(Ae^{i\mathbf{r}\cdot\boldsymbol{\sigma}t})\cdot\boldsymbol{\sigma}|_{t=0}\\&=\frac{d}{dt}Ae^{i\mathbf{r}\cdot\boldsymbol{\sigma}t}\sigma_3e^{-i\mathbf{r}\cdot\boldsymbol{\sigma}t}A^{*}|_{t=0}\\&=A[i\mathbf{r}\cdot\boldsymbol{\sigma},\sigma_3]A^{*}=A(-2(\mathbf{r}\times\mathbf{e}_3)\cdot\boldsymbol{\sigma})A^{*}\\&=2A(\mathbf{e}_3\times\mathbf{r})\cdot\boldsymbol{\sigma}A^{*}=2H(A)(\mathbf{e}_3\times\mathbf{r})\cdot\boldsymbol{\sigma}\end{aligned}$$

thus,

$$\pi_{*}(Ai\mathbf{r}\cdot\boldsymbol{\sigma})=2H(A)(\mathbf{e}_3\times\mathbf{r}) \qquad (**)$$

Let $\mu\in\Lambda^{2}(S^2,\mathbb{R})$ be defined at $H(A)\mathbf{e}_3$ by

$$\mu\big(H(A)(\mathbf{e}_3\times\mathbf{r}),H(A)(\mathbf{e}_3\times\mathbf{r}')\big)=(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{e}_3$$

Note that μ is the area element of S^2. From (*) and (**), we have

$$\Omega = \frac{i}{2}(\pi^*\mu)\sigma_3$$

The *Chern class* of a $U(1)$ bundle $\pi: P \to M$ with connection ω is the de Rham cohomology class $c(P)$ of $(i/2\pi)\Omega$, where Ω is the unique 2-form on M such that $\pi^*\Omega = \Omega$. Considering $\pi: SU(2) \to S^2$ as a $U(1)$ bundle, we have shown $\Omega = (i/2)\mu$, whence for $P = SU(2)$,

$$c(P)[S^2] \equiv \frac{i}{2\pi}\int_{S^2}\Omega = \frac{i}{2\pi}\cdot\frac{i}{2}\int_{S^2}\mu = -1$$

We can obtain a Chern number $+1$ by reversing the direction of the $U(1)$ action on $SU(2)$. More generally, if we let $\mathbb{Z}_k$ be the cyclic subgroup of $U(1)$ of order k, then there is a $U(1)$ action on the quotient space $P_k \equiv SU(2)/\mathbb{Z}_k$ (called a Lens space); the action is defined by letting $e^{i\theta}$ act on P_k via multiplication by $e^{-i(\theta/k)\sigma_3}$. Then $c(P_k)[S^2]$ is calculated to be k. In general, whenever the Chern class of a $U(1)$ bundle $\pi: P \to M$ is evaluated on a closed surface S in M the resulting "Chern number" $c(P)[S]$ is always an integer. The Chern class (and Chern numbers) are actually independent of the choice of the connection ω on P (e.g., see Bleecker, 1981, and Kobayashi and Nomizu, 1969), and hence depend only on the bundle; i.e., Chern numbers are "topological quantum numbers." In a physical context, when M is a space-time (possibly, topologically nontrivial), the electromagnetic "4-vector potential" A (relative to a choice of gauge $\tau: U \to P$) is related to ω by $\tau^*\omega = (ie/\hbar c)A$. Consequently, on U we have

$$\Omega = d(\tau^*\omega) = \frac{ie}{\hbar c}\,dA = \frac{-ie}{\hbar c}F$$

and since Ω and the electromagnetic field strength F are globally defined, $\Omega = (-ie/\hbar c)F$ throughout M. Let S be a closed surface in M. The magnetic charge e' enclosed by S is $(1/4\pi)\int_S F$, and thus we have

$$c(P)[S] = \frac{i}{2\pi}\int_S \Omega = \frac{e}{2\pi\hbar c}\int_S F = \frac{2ee'}{\hbar c}$$

Hence, the integrality of Chern numbers is equivalent to Dirac's famous quantization condition on the electric charge of a particle in the presence of a magnetic monopole. An examination of Dirac's 1931 paper (Dirac, 1931) reveals a nice proof of the integrality of Chern numbers for $U(1)$ bundles, before they were considered by Chern (1946) in a more general context.

3. NATURAL OPERATORS ON EQUIVARIANT FUNCTIONS

For a principal G bundle $\pi: P \to M$ and a representation $r: G \to GL(W)$, we define $\overline{\Lambda}^k(P, W)$ to be the subspace of all $\alpha \in \Lambda^k(P, W)$ such that

$$(1) \quad \alpha(A^*, \cdot, \ldots, \cdot) = 0, \qquad \text{for all } A \in \mathrm{G}, \textit{ and}$$

$$(2) \quad R_g^* \alpha = r(g^{-1})\alpha, \qquad \text{for all } g \in G.$$

These are called *horizontal, equivariant W-valued k-forms*; when $k = 0$, condition (1) is dropped and (2) becomes $\alpha(pg) = r(g^{-1})\alpha(p)$ for all $p \in P$ and $g \in G$.

When $W = \mathrm{G}$ and $r: G \to GL(\mathrm{G})$ is the adjoint representation ($r(g)A = gAg^{-1}$), we do *not* have $\omega \in \overline{\Lambda}^1(P, \mathrm{G})$ for a connection ω because $\omega(A^*) = A$ in violation of (1). However, if ω' is another connection, we have $\omega - \omega' \in \overline{\Lambda}^1(P, \mathrm{G})$. The operation D^ω, introduced in Section 2, preserves conditions (1) and (2), and so $D^\omega: \overline{\Lambda}^k(P, W) \to \overline{\Lambda}^{k+1}(P, W)$. Even though $\omega \notin \overline{\Lambda}^1(P, \mathrm{G})$, it is true that $\Omega \equiv D^\omega \omega \in \overline{\Lambda}^2(P, \mathrm{G})$, and the Bianchi identity says $D^\omega \Omega = 0$. One can prove (e.g., see Bleecker, 1981, p. 44) that $D^\omega \alpha = d\alpha + r'(\omega) \wedge \alpha$, where

$$(r'(\omega) \wedge \alpha)(X_1, \ldots, X_{k+1}) = \frac{1}{k!} \sum_\sigma (-1)^\sigma r'(\omega(X_{\sigma_1}))\alpha(X_{\sigma_2}, \ldots, X_{\sigma_{k+1}})$$

the sum being over all permutations σ of $\{1, \ldots, k+1\}$. D^ω is called the *exterior covariant differentiation operator*, and it is related to the "principle of minimal coupling," as follows. Let $k = 0$ and suppose $\tau: U \to P$ is a local section. For $\tilde{\Psi} \in \overline{\Lambda}^0(P, W)$, let $\Psi = \tau^* \tilde{\Psi} = \tilde{\Psi} \circ \tau: U \to W$, and set $\mathsf{A} = \tau^* \omega \in \Lambda^1(U, \mathrm{G})$. Then $\tau^*(D^\omega \tilde{\Psi}) = \tau^*(d\tilde{\Psi} + r'(\omega)\tilde{\Psi}) = d\Psi + r'(\mathsf{A})\Psi$. As is customary, this expression is complicated by adorning it with indices relative to a coordinate system x^ν on U, a basis of W, and a basis $\{T_i\}$ of G; we then get

$$\tau^*(D^\omega \tilde{\Psi})_\nu^a = \frac{\partial \Psi^a}{\partial x^\nu} + \mathsf{A}_\nu^i r'(T_i)_b^a \Psi^b$$

This shows that the covariant derivative formed through minimal coupling is essentially horizontal differentiation via D^ω on P.

To make $\overline{\Lambda}^k(P, W)$ a Hilbert space, we assume $r: G \to U(W)$ is unitary relative to some Hermitian scalar product $\hat{h}$ on W, and let h be a metric tensor on M. Define a pairing $\langle , \rangle : \overline{\Lambda}^k(P, W) \times \overline{\Lambda}^k(P, W) \to C^\infty(P)$ by

$$\langle \alpha, \beta \rangle = \frac{1}{k!} \hat{h}_{ab} h^{i_1 j_1} \cdots h^{i_k j_k} \alpha^a{}_{i_1 \cdots i_k} \beta^b{}_{j_1 \cdots j_k}$$

By equivariance, $\langle \alpha, \beta \rangle$ is constant on fibers, and hence may be regarded as a function on M. We set $(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \mu$, where μ is the volume element on M relative to h. The scalar product $(,)$ is positive definite when $k = 0$ or h is Riemannian, in which case we could complete $\overline{\Lambda}^k(P, W)$ to a Hilbert space. Now $D^\omega : \overline{\Lambda}^k(P, W) \to \overline{\Lambda}^{k+1}(P, W)$ has a formal adjoint $\delta^\omega : \overline{\Lambda}^{k+1}(P, W) \to \overline{\Lambda}^k(P, W)$ such that $(D^\omega \alpha, \gamma) = (\alpha, \delta^\omega \gamma)$; δ^ω is called the *covariant codifferential*. For a local section $\tau: U \to P$ and for $\Psi = \tau^* \tilde{\Psi}$ and $\mathsf{A} = \tau^* \omega$, we have

$$-\tau^*(\delta^\omega \tilde{\Psi})_{\nu_1 \cdots \nu_k} = \Psi^\nu{}_{\nu_1 \cdots \nu_k | \nu} + \mathsf{A}^i_\nu r(T_i) \Psi^\nu{}_{\nu_1 \cdots \nu_k}$$

where "|" denotes covariant differentiation relative to the Levi-Cività connection of h, and the W indices of Ψ are omitted. Note that the source-free Yang–Mills equation is simply $\delta^\omega \Omega = 0$.

A formally self-adjoint operator (the *covariant Hodge Laplacian*) is defined by $\Delta^\omega = \delta^\omega D^\omega + D^\omega \delta^\omega : \overline{\Lambda}^k(P, W) \to \overline{\Lambda}^k(P, W)$; note that Δ^ω and δ^ω depend on the metric h on M as well as ω on P. We will confine ourselves to the case $k = 0$, and use the notation $C(P, W) \equiv \overline{\Lambda}^0(P, W) = \{\alpha : P \to W \text{ such that } \alpha(pg) = r(g)^{-1}\alpha(p)\}$. Then Δ^ω on $C(P, W)$ is $\delta^\omega D^\omega$, since $\delta^\omega = 0$ on $C(P, W)$. Roughly, $C(P, W)$ is the quantum mechanical state space of a spin-0 (multicomponent) particle, and as observables,

$$\frac{\hbar^2}{2m} \Delta^\omega \sim \text{kinetic energy}, \qquad \text{if } \dim M = 3$$

$$\frac{\hbar^2}{c^2} \Delta^\omega \sim (\text{mass})^2, \qquad \text{if } \dim M = 4$$

For $\tilde{\Psi} \in C(P, W)$, $\tau: U \to P$, $\Psi = \tau^* \tilde{\Psi}$, and $\mathsf{A} = \tau^* \omega$, we have the familiar expression

$$\tau^*(\Delta^\omega \tilde{\Psi}) = -h^{\mu\nu}\left(D_\mu + \mathsf{A}^i_\mu r(T_i)\right)\left(D_\nu + \mathsf{A}^j_\nu r(T_j)\right)\Psi$$

where D_μ is covariant differentiation (relative to the Levi-Cività connection of h) in the direction of $\partial / \partial x^\mu$. Now, Δ^ω has a nice characterization as the "horizontal Laplacian" of a generalized Kaluza–Klein metric on P which we review in the next section. The close relation between the spectrum of Δ^ω

on $C(P, W)$ and that of the usual Laplace operator on P (relative to the Kaluza–Klein metric) was exploited in Bleecker (1983). Our emphasis will be somewhat different here, in that we will concentrate eventually on the effect of infinitesimal changes of ω and h on the eigenvalues of Δ^ω. Moreover, the relation between symmetries of ω and h and degeneracy of the eigenvalues will be examined.

4. GENERALIZED KALUZA–KLEIN METRICS AND Δ^ω

Let $\hat{k}$ be any Ad-invariant scalar product on G; i.e., $\hat{k}(gAg^{-1}, gBg^{-1}) = \hat{k}(A, B)$. Given a connection ω for the G bundle $\pi: P \to M$ and a metric tensor h on M, we define the *generalized Kaluza–Klein metric* $\bar{h}$ on P by

$$\bar{h}(X, Y) = \hat{k}(\omega(X), \omega(Y)) + h(\pi_* X, \pi_* Y)$$

We collect some interesting properties of $\bar{h}$:

(1) $R_g^* \bar{h} = \bar{h}$; i.e., R_g is an isometry of $(P, \bar{h})$.

(2) If $\gamma(t)$ is a geodesic on P relative to $\bar{h}$, then $\omega(\gamma'(t))$ is a constant matrix $Q \in$ G.

(3) The projection $\pi(\gamma(t))$ of the geodesic onto M is the path of a charged particle (of mass m and "generalized charge" $q \equiv Q/m$) which is subject to the field strength of Ω under the Lorentz force law. In other words,

$$m \frac{D}{dt}(\pi \circ \gamma)'(t) = \hat{k}\left(q, \Omega\left(\gamma'(t), \pi_*^{-1}(\cdot)\right)\right)^{\#}$$

where the left side is mass times the covariant "4-acceleration" of $\pi \circ \gamma$ and the sharp on the right side indicates that indices are raised.

(4) The scalar curvatures, R_P of $\bar{h}$ at $p \in P$, R_M of h at $\pi(p)$, and R_G of G relative to the bi-invariant metric on G induced by k, are related by

$$R_P = R_M + R_G - 1/2\langle \Omega, \Omega \rangle$$

Hence, setting the first variation with respect to h of $\int_M R_P \mu$ equal to zero, one obtains the Einstein field equations with a cosmological term (proportional to R_G) and with source arising from Ω; the variation of $\int_M R_P \mu$ with respect to ω, yields the Yang–Mills equation $\delta^\omega \Omega = 0$.

(5) If $X \in H_p \equiv \{X \in T_p P: \omega(X) = 0\}$, and $A \in$ G, then $\text{Ric}_P(X, A^*) = -(1/2)\hat{k}(\delta^\omega \Omega(X), A)$.

Consequently, the Yang–Mills equation $\delta^\omega \Omega = 0$ holds exactly when the horizontal and vertical subspaces of $T_p P$ are orthogonal relative to the Ricci tensor Ric_P of $\bar{h}$ at all $p \in P$.

Most of these results are stated in many articles; they are all proved in Bleecker (1981). Of particular relevance to us here, are various Laplace operators on P, which we now define.

Let $\gamma_1, \ldots, \gamma_n$ be a "frame" of horizontal geodesics through $p \in P$; i.e., $\gamma_1'(0), \ldots, \gamma_n'(0)$ is an orthonormal basis of H_p. For α in $C^\infty(P, W) \equiv$ the space of smooth W-valued functions on P (W any vector space of finite dimension), we define $\Delta^H : C^\infty(P, W) \to C^\infty(P, W)$ by

$$(\Delta^H \alpha)_p = - \sum_{\nu=1}^{n} \frac{d^2}{dt^2} \alpha(\gamma_\nu(t))$$

Define Δ^V similarly, using frames of vertical geodesics. Then $\Delta \equiv \Delta^H + \Delta^V$ is (minus) the usual Laplacian on $C^\infty(P, W)$ relative to $\bar{h}$. Now, let $r: G \to U(W)$ be a unitary representation. The following facts are proven in Bleecker (1983):

(1) $\Delta^\omega = \Delta^H | C(P, W) \equiv$ restriction of Δ^H to $C(P, W)$.

(2) Δ^H, Δ^V, and Δ are mutually commuting.

(3) Let r be irreducible with Casimir operator $-\sum_i r(T_i)^2 = c_r I : W \to W$, where $\{T_i\}$ is an orthonormal basis of G relative to $\hat{k}$. Then $\Delta^V \alpha = c_r \alpha$ for all $\alpha \in C(P, W)$; i.e., $\Delta^V = c_r I : C(P, W) \to C(P, W)$. Hence, on $C(P, W)$, $\lambda \in \operatorname{spec}(\Delta^\omega) \Leftrightarrow \lambda + c_r \in \operatorname{spec}(\Delta)$; i.e., $\operatorname{spec}(\Delta^\omega) = \operatorname{spec}(\Delta) - c_r$.

The essential point of (3) is that in many circumstances, $\operatorname{spec}(\Delta)$ is easier to compute than $\operatorname{spec}(\Delta^\omega)$. We illustrate this in a simple case.

Example 4. For $\pi: SU(2) \to S^2$, we found in Example 3 that $\pi_*: T_A SU(2) \to T_{\pi(A)} S^2$ is given by $\pi_*(A i r \cdot \sigma) = 2H(A)(\mathbf{e}_3 \times \mathbf{r})$. Thus, $A(i/2)\sigma_1$ and $A(i/2)\sigma_2$ project to the orthonormal frame $H(A)(\mathbf{e}_2)$ and $-H(A)(\mathbf{e}_1)$. Choosing $\hat{k}$ on $U_3(1)$ so that $\hat{k}[i/2\sigma_3, (i/2)\sigma_3] = 1$, the Kaluza–Klein metric $\bar{h}$ on $SU(2)$ is the bi-invariant metric $\bar{h}(A(i/2)\mathbf{r} \cdot \sigma, A(i/2)\mathbf{r}' \cdot \sigma) = \mathbf{r} \cdot \mathbf{r}'$. Let $r_N : U_3(1) \to U(\mathbb{C})$ be given by $r_N(e^{i\theta\sigma_3})z = e^{iN\theta}z$, $N = 0, \pm 1, \pm 2, \ldots$. We write $\mathbb{C}$ as $\mathbb{C}_N$ when regarding it as the representation space of r_N. Thus, $C(SU(2), \mathbb{C}_N)$ consists of all maps $\alpha: SU(2) \to \mathbb{C}_N$ such that $\alpha(Ae^{i\theta\sigma_3}) = e^{-iN\theta}\alpha(A)$. The curves $\theta \mapsto Ae^{1/2\theta\sigma_j}$ ($j = 1, 2, 3$) form a frame of geodesics through A in $SU(2)$ (vertical for $j = 3$, and horizontal for $j = 1, 2$). Thus

$$\begin{aligned} \Delta^V \alpha(A) &= -\frac{d^2}{d\theta^2} \alpha(Ae^{(i/2)\theta\sigma_3}) \Big|_{\theta=0} \\ &= -\frac{d^2}{d\theta^2} e^{-(i/2)N\theta} \alpha(A) \Big|_{\theta=0} = \left(\frac{N}{2}\right)^2 \alpha(A) \end{aligned}$$

i.e., $\Delta^V = (N/2)^2 I$ on $C(SU(2),\mathbb{C}_N)$, in accordance with fact (3) above. To determine the eigenvalues of Δ on $C^\infty(SU(2),\mathbb{C})$, we follow the Peter–Weyl procedure. Let $D^j: SU(2) \to U(W^j)$ be the spin-j representation; $\dim W^j = 2j+1$, $j = 0, 1/2, 1, \ldots$. Let $\{e^j{}_m\}$ $(m = -j, -j+1, \ldots, j-1, j)$ be an orthonormal basis for W^j which is standard in the sense that $D^j(e^{(i/2)\theta\sigma_3})e^j{}_m = e^{im\theta}e^j{}_m$. Define $D^j{}_{nm}: SU(2) \to \mathbb{C}$ by $D^j{}_{nm}(A) = \langle D^j(A)e^j{}_m, e^j{}_n\rangle$. Then the complex conjugate function $\overline{D}^j{}_{nm}$ is in $C(SU(2),\mathbb{C}_{2m})$, and it is well known that $\Delta\overline{D}^j{}_{nm} = j(j+1)\overline{D}^j{}_{nm}$. The Peter–Weyl theorem implies that $\{\overline{D}^j{}_{nm}: j = 0, 1/2, 1, \ldots;\ m, n = -j, \ldots, j\}$ is a complete orthonormal set of eigenfunctions of Δ. The eigenfunctions of $\Delta^\omega = \Delta - \Delta^V$ on $C(SU(2),\mathbb{C}_N)$ are then

$$\overline{D}^j{}_{nN/2} \quad j = \left|\frac{N}{2}\right|, \left|\frac{N}{2}\right| + 1, \left|\frac{N}{2}\right| + 2, \ldots$$

$$n = -j, -j+1, \ldots, j-1, j$$

with eigenvalues $j(j+1) - |N/2|^2$ of multiplicity $2j+1$. For $N = 0$, these are constant on the fibers of $\pi: SU(2) \to S^2$, and in fact induce the usual spherical harmonics on S^2. For $N \neq 0$, the functions $\overline{D}^j_{nN/2}$ are not constant on fibers, and thus do not induce $\mathbb{C}$-valued functions on S^2. However, they can be regarded as sections of the associated complex line bundle $V_N \equiv SU(2) \times \mathbb{C}_N / U_3(1) \to S^2$. Indeed there is a natural correspondence

$$\begin{array}{ccc} \Gamma(S^2, V_N) & \leftrightarrow & C(SU(2),\mathbb{C}_N) \\ \text{(sections)} & & \text{(equivariant functions)} \end{array}$$

Wu and Yang (1976) call these sections "monopole harmonics," since they may be used to expand the wave function (actually a section) of a particle of charge Ne in the field of a monopole; see also Greub and Petry [6].

5. AUTOMORPHISMS, DEGENERACY, AND SPLITTING

In the preceding Example 4, the eigenspaces of Λ^ω on $C(SU(2),\mathbb{C}_N)$ are degenerate (except for $N = 0$, $j = 0$). When $N = 0$, this degeneracy is easily explained since $O(3)$ acts on the spherical harmonics. For $N \neq 0$, the symmetry responsible for the degeneracy is more obscure. For this reason, we turn to the study of automorphisms of G bundles.

An *automorphism* of a G bundle $\pi: P \to M$ is a diffeomorphism $F: P \to P$ such that $F(pg) = F(p)g$. F "covers" a diffeomorphism $\tilde{F}: M \to M$, uniquely determined by the property $\tilde{F}(\pi(p)) = \pi(F(p))$. Let

Aut(P) be the group (under composition) of all automorphisms of P. Then the subgroup $GA(P) \equiv \{ F \in \text{Aut}(P) : \tilde{F} = I \}$ is called the *group of gauge transformations*. It is important to note that $R_g : P \to P$ is *not* a gauge transformation unless g is in the center of G; indeed, $g \in$ center of $G \Leftrightarrow gg' = g'g$ for all $g' \in G \Leftrightarrow R_g(pg') = pg'g = pgg' = R_g(p)g'$ for all $p \in P$, $g' \in G$. Now, Aut(P) acts on $\mathrm{C}(P) \equiv$ the set of all connections on P via $F \cdot \omega \equiv F^{-1*}\omega$. Also, Aut($P$) acts on $\overline{\Lambda}^k(P, W)$ via $F \cdot \alpha = F^{-1*}\alpha$; proofs are found in Bleecker (1981). For $\omega \in \mathrm{C}(P)$ and h a metric tensor on M, set

$$\text{Aut}(P, \omega, h) \equiv \left\{ F \in \text{Aut}(P) : F \cdot \omega = \omega, \tilde{F}^* h = h \right\}$$

Note that Aut(P, ω, h) is finite dimensional, since it is contained in the isometry group of the Kaluza–Klein metric $\bar{h}$ associated with ω and h; the reverse inclusion does not hold in general, since R_g is always an isometry of $\bar{h}$. From the fact that Aut(P, ω, h) preserves $\bar{h}$ (and hence sends geodesics to geodesics), it is easy to check that for $\alpha \in C(P, W)$ (or, more generally, $\alpha \in \overline{\Lambda}^k(P, W)$) and $F \in \text{Aut}(P, \omega, h)$, we have $\Delta^\omega(F \cdot \alpha) = F \cdot \Delta^\omega \alpha$. Thus, Aut($P, \omega, h$) leaves the eigenspaces of Δ^ω invariant, and we have the important fact: The eigenspaces of Δ^ω on $\overline{\Lambda}^k(P, W)$ are representation spaces of the group Aut(P, ω, h).

Example 5. Consider $\pi : SU(2) \to S^2$ with the standard connection ω of Example 2 and usual metric h on S^2. The isometries of $SU(2)$ (with the bi-invariant Kaluza–Klein $\bar{h}$ of Example 4) are of the form $A \mapsto BAC$ or $A \mapsto BA^{-1}C$, where $A, B, C \in SU(2)$. The latter is not in Aut(P), while the former is in Aut(P, ω, h) iff $C \in U_3(1)$. Thus, $\text{Aut}(P, \omega, h) \cong SU(2) \dot{\times} U_3(1)$, where the dot indicates that we do not have a true direct product because of a discrete Z_2 intersection; i.e., $B = -I$, $C = I$ and $B = I$, $C = -I$ give the same automorphism $A \mapsto -A$. The $(2j+1)$-dimensional eigenspace of Δ^ω on $C(SU(2), C_N)$ with eigenvalue $j(j+1) - |N/2|^2$ is a representation space of Aut(P, ω, h), namely, (spin j)$\otimes r_{-N}$. Note that the Z_2 intersection forces $2j$ and N to have the same parity.

We can try to "split" an isolated degenerate eigenvalue λ of Δ^ω on $C(P, W)$ by perturbing ω to $\omega' = \omega + \tau$ for some $\tau \in \overline{\Lambda}^1(P, \mathrm{G})$. Indeed, for generic τ, we expect that Aut(P, ω', h) will be trivial if G is centerless. To find the splitting of λ to first order in τ, we use the easily derived formula

$$\Delta^{\omega'}\alpha = \Delta^\omega \alpha + \delta^\omega(r(\tau)\alpha) - r(\tau) \cdot D^\omega \alpha - r(\tau) \cdot r(\tau)\alpha,$$

where $r(\tau) \cdot D^\omega \alpha = \sum \bar{h}^{ij} r(\tau(E_i)) D^\omega \alpha(E_j)$ at p for any basis $E_1, \ldots, E_n$ of $H_p \subset T_p P$. From standard first-order perturbation theory (e.g., see Kato, 1966) the "split" eigenvalues of $\Delta^{\omega'}$ about λ are given (to first order in τ) by

$\lambda + \mu_i$ where the μ_i are eigenvalues of the following quadratic form on the eigenspace $V(\lambda)$ of λ:

$$Q(\alpha) = (\delta^{\omega}(r(\tau)\alpha) - r(\tau)\cdot D^{\omega}\alpha, \alpha)$$
$$= 2(r(\tau)\alpha, D^{\omega}\alpha), \qquad \alpha \in V(\lambda)$$

This type of splitting via perturbing the gauge potential is the natural generalization of the Zeemann or Stark effects to the case of arbitrary G bundles. Could it be that the observed mass splittings in various multiplets are due to some deviation of a connection ω (for a G bundle over the universe) from a more "symmetrical" connection ω_0 with Aut(P, ω_0, h) larger than Aut(P, ω, h)? Perhaps there are other domains within the universe (or other universes) where the deviation is different. We have more to say about this in Section 6.

The eigenvalues of Δ^{ω} on $C(P, W)$ are also sensitive to changes in the metric h on M (i.e., gravity). The quadratic form that gives the first-order splitting of the eigenvalues λ under the perturbation $h \to h + s$, where s is a "small" (relative to h) symmetric 2-tensor is given by

$$Q(\alpha) = -\int_M s^{\mu\nu}\left[\hat{h}(D_{\mu}^{\omega}\alpha, D_{\nu}^{\omega}\alpha) + \frac{1}{4}\Delta(|\alpha|^2)h_{\mu\nu}\right]\mu_h$$

where $\Delta(|\alpha|^2)$ is the Laplacian of the function $|\alpha|^2 \equiv \hat{h}(\alpha, \alpha)$ on M relative to the metric h. The derivation of this formula is not very hard. It hopefully will appear in a later publication, along with similar formulas for the case of eigenvalues of Dirac operators on bispinor fields with co-efficients in vector bundles associated to $\pi: P \to M$. Interestingly, the quadratic form on an eigenspace of the Dirac operator (for a perturbation of the connection) looks like a sum of Yukawa terms. Normally, such terms are unattractively introduced via Higgs fields, but here the role of the Higgs fields is taken over by the deviation of the connection from one of greater symmetry.

6. DECREASING SYMMETRY BY INCREASING HOLONOMY

In the notation of Section 5, we set $GA(P, \omega) \equiv \{F \in GA(P): F^*\omega = \omega\}$ = the group of gauge transformations preserving ω. Note that $GA(P, \omega) = GA(P) \cap \text{Aut}(P, \omega, h) \subseteq \text{Aut}(P, \omega, h)$, whence $GA(P, \omega)$ also acts on the eigenspaces of Δ^{ω} on $\overline{\Lambda}^k(P, W)$. We will exhibit an explicit isomorphism from $GA(P, \omega)$ to the centralizer in G of the holonomy group of ω, which is defined as follows. Fix a point $p \in P$ and let P_0 be the set of all points q of P that can be joined to p by a smooth curve γ, all of whose tangent vectors are horizontal (i.e., $\omega(\gamma'(t))=0$). Let $G_0=\{g\in G: pg\in P_0\}$ (Recall, p is fixed). P_0 turns out to be a (immersed) submanifold of P, and G_0 is a subgroup of G. Indeed, $\pi: P_0\to M$ is a (reduced) subbundle of $\pi: P\to M$, and is called the *holonomy bundle* of ω through p, while G_0 is the *holonomy group* of ω at p; see Kobayashi and Nomizu (1963) for details.

For $F\in GA(P)$, let $g_F\in G$ be such that $F(p)=pg_F$ (p is still fixed). Define $\Phi: GA(P,\omega)\to G$ by $\Phi(F)=g_F$.

Proposition: $\Phi: GA(P,\omega)\to G$ maps $GA(P,\omega)$ isomorphically onto the centralizer $C(G_0)$ of G_0 (i.e., $\Phi: GA(P,\omega)\cong C(G_0)\equiv\{g\in G: gg_0=g_0g$ for all $g_0\in G_0\}$).

Proof. Let $F\in GA(P,\omega)$ and $g_0\in G_0$. To show $\Phi(F)\equiv g_F\in C(G_0)$, we need to prove $g_Fg_0=g_0g_F$. Let γ be a horizontal curve connecting p to pg_0 and let $\bar\gamma=\pi\circ\gamma$. Since R_{g_F} preserves horizontality, we have $R_{g_F}\circ\gamma$ is a horizontal curve connecting pg_F to pg_0g_F. Now, as F is a gauge transformation preserving ω (and hence horizontality), we have that $F\circ\gamma$ is a horizontal curve that connects pg_F to $F(pg_0)=F(p)g_0=pg_Fg_0$. Now $R_{g_F}\circ\gamma$ and $F\circ\gamma$ are both horizontal lifts of $\bar\gamma$ beginning at pg_F. Thus, by uniqueness of horizontal lifts, the end points of $F\circ\gamma$ and $R_{g_F}\circ\gamma$ agree; i.e., $pg_Fg_0=pg_0g_F$. Thus, $g_Fg_0=g_0g_F$, and we have $\Phi(GA(P,\omega))\subset C(G_0)$. We prove that Φ is one to one; i.e., $g_F=g_{F'}\Rightarrow F=F'$. Let $q\in P$. We must prove $F(q)=F'(q)$. Let $\bar\gamma$ be a curve joining $\pi(p)$ to $\pi(q)$ and let γ be the horizontal lift of $\bar\gamma$ joining p to $q_0\in\pi^{-1}(\pi(q))$, say $q=q_0g$. Now $F\circ\gamma$ and $F'\circ\gamma$ are both horizontal lifts of γ beginning at $pg_F=pg_{F'}$. Thus, the end points $F(q_0)$ and $F'(q_0)$ are equal. Thus, $F(q)=F(q_0g)=F(q_0)g=F'(q_0)g=F'(q_0g)=F'(q)$, as required; i.e., Φ is one-to-one. Finally, we prove $\Phi(GA(P,\omega))\supset C(G_0)$. Let $\tilde g\in C(G_0)$. For $q\in P$ with $q=q_0g$, $q_0\in P_0$, define $F: P\to P$ by $F(q)=q_0\tilde gg$. Of course, we need to prove F is well defined and $F\in GA(P,\omega)$; then $\Phi(F)=\tilde g$, and we will be done. Suppose $q=q_0'g'$ where $q_0'\in P_0$ and $g'\in G$. Then $q_0\tilde gg=qg^{-1}\tilde gg=q_0'(g'g^{-1})\tilde gg=q_0'\tilde g(g'g^{-1})g=q_0'\tilde gg'$ (i.e., F is well defined), since $\tilde g\in C(G_0)$ and $g'g^{-1}\in G_0$. For $g_1\in G$, we have $qg_1=q_0gg_1$ whence $F(qg_1)=q_0\tilde ggg_1=F(q)g_1$ [i.e., $F\in GA(P)$]. To show $F\in GA(P,\omega)$, we need to show that F_* maps horizontal subspaces to horizontal subspaces. On P_0, F coincides with $R_{\tilde g}$ which we know preserves horizontal subspaces. Since $H_{q_0}\subset T_{q_0}P_0$ for any $q_0\in P_0$, we then have $F_*H_{q_0}=R_{\tilde g*}H_{q_0}=H_{F(q_0)}$. Thus, for arbitrary $q\in P$ with $q=q_0g$ ($g_0\in P_0$), we have $F_*H_q=F_*R_{g*}H_{q_0}=(F\circ R_g)_*H_{q_0}=(R_g\circ F)_*H_{q_0}=R_{g*}\circ F_*H_{q_0}=R_{g*}H_{F(q_0)}=H_{F(q_0)g}=H_{F(q)}$.

Note that as G_0 increases, $GA(P,\omega)\cong C(G_0)$ decreases; thus, the eigenvalues of Δ^ω on $\overline{\Lambda}^k(P,\omega)$ may undergo progressive splittings as the holonomy group of a variable ω increases to larger subgroups of G.

The case $G_0 = \{I\}$ admits an easy, yet instructive, analysis. In this case, P is trivial (say, $P = M \times G$). The submanifolds $M \times \{g\}$ have horizontal tangent spaces and are totally geodesic relative to the Kaluza–Klein metric induced by a metric tensor h on M, ω, and $\hat{k}$ on G. Let $\Psi \in C^\infty(M)$ be an eigenfunction of the Laplace operator for h on M; say, $\Delta\Psi = \lambda\Psi$. Let $r: G \to GL(W)$ be a representation and select any $w \in W$. Define $\Psi_w \in C(P, W)$ by $\Psi_w(x, g) = \Psi(x)r(g^{-1})(w)$. Then $\Delta^\omega \Psi_w = \Delta^H \Psi_w = \lambda$. Indeed, every eigenfunction of Δ^ω on $C(P, W)$ is of this form (when $G_0 = \{I\}$). Thus, the eigenvalues of Δ^ω on $C(P, W)$ are simply the (r-independent!) eigenvalues λ of Δ on M, and the multiplicity of λ for Δ^ω on $C(P, W)$ is dim(W) times the multiplicity of λ for Δ on $C^\infty(M)$. Only when $G_0 \neq \{I\}$, is there any chance that the eigenvalues of Δ^ω will depend on the representation r; in Example 4 of Section 4, the holonomy group is $U_3(1)$ and the eigenvalues of Δ^ω on $C(SU(2), \mathbb{C}_N)$ are seen to depend on N.

It is tempting to conjecture that the three (or more) generations of elementary quarks and leptons which transform identically with respect to grand unification groups [e.g., $SU(5)$, $SO(10)$, etc.] actually originated from an eigenvalue of Δ on $C^\infty(M)$ (of multiplicity 3 or more) at a stage when a hypothetical connection ω was closer to having trivial holonomy. In this way, gravitational aspects (e.g., h on M) may have played an essential role in producing multiple generations.

There are some important observations that must be made with regard to the relation between mass degeneracy and the size of the grand unification group, say, G. Usually, masses are introduced by adding mass terms to a Lagrangian that possesses local gauge invariance relative to G. These masses (possibly 0) are split by the successive introduction of Higgs fields that have vacuum expectations invariant under a corresponding decreasing sequence of subgroups of G. The Yukawa terms, involving the original fundamental fields (i.e., leptons and quarks) and the Higgs fields, are responsible for giving new mass terms (for the original fields) which are invariant under the smaller group. In this way, the masses of the original fields may lose their degeneracy. While the Higgs mechanism has many useful features (e.g., it permits renormalizability even though the broken gauge fields acquire mass, and the Goldstone modes disappear), there is a feeling that their function is primarily to parametrize our ignorance; also, there is the unnatural fine-tuning problem in connection with the hierarchy phenomenon. The viewpoint that we take in this paper is that masses are not generated or split by adding terms to the Lagrangian of the universe, but rather that the masses are given via the eigenvalues of Δ^ω (or more realistically, of the associated covariant Dirac operators) on the spaces $C(P, W)$, for a given connection ω that exists as a classical field on P (perhaps, highly confined) in the same sense as h on M. If we take this point of view, then as we have seen, mass degeneracy (i.e., the degeneracy of the eigenvalues of Δ^ω on $C(P, W)$) does not necessarily follow from the size of G, but rather the size of $GA(P, \omega) \cong C(G_0)$, where G_0 is the holonomy group of ω. In other words, $GA(P, \omega)$ (*not* G) acts on the eigenspaces. Indeed, for a generic connection ω, it should be possible to prove (e.g., see Kobayashi and Nomizu, 1963, p. 90) that G_0 is typically all of G, whence $GA(P, \omega) \cong C(G)$ is the Abelian center of G; then, since the irreducible representations of Abelian groups are one dimensional, we expect that the eigenspaces of Δ^ω will be one dimensional (i.e., there will be no mass degeneracy generically, regardless of the size of G). Of course, $GA(P, \omega) \cong C(G)$ cannot be larger than G; so a large G permits many degeneracies, but it does not force them. Relatively small mass splittings are observed within various well-defined multiplets. In the present context, this means that the (most likely, generic) connection ω for the actual universe is not too far from a nongeneric connection ω_1 with a smaller holonomy group; indeed, ω_1 may then be close to some ω_2 with even smaller holonomy, and so on. (This may explain why there appear to be multiplets within multiplets.)

One may ask how ω got to be where it is. Perhaps the "answer" lies in the anthropomorphic principle; ω is generic enough to permit the existence of nondegenerate building blocks to construct interesting structures, but ω is not so far from nongenericity that chaos results. There may be many other universes with different geometries and connections, but creatures like us may inhabit only a chosen few.

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